Combinatorics

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These notes have not been checked by Prof. A.G. Thomason and should not be regarded as official notes for the course. In particular, the responsibility for any errors is mine — please email Sebastian Pancratz (`sfp25`) with any comments or corrections.

Contents

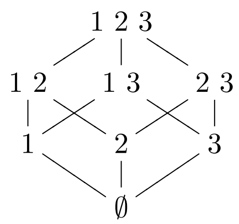
Notation

We denote $[n] = \{1, \ldots, n\}$ and more generally write $[m, n] = \{m, \ldots, n\}$ for $m, n \in \mathbb{N}$. Given a set X, we write $\mathcal{P}(X) = \{Y : Y \subset X\}$ or sometimes $\mathcal{P}X$ for the power set. Further, we write $X^{(r)} = \{Y \subset X : |Y| = r\}$ and call a family $\mathcal{F} \subset \mathcal{P}X$ *r-uniform* if $\mathcal{F} \subset X^{(r)}$. We say $\mathcal{F}$ is *uniform* if it is r-uniform for some r.

Chapter 1

Antichains

A family $\mathcal{A} \subset \mathcal{P}X$ is a *chain* if whenever $A, B \in \mathcal{A}$ then $A \subset B$ or $B \subset A$. Similarly, $\mathcal{A} \subset \mathcal{P}X$ is an *antichain* if whenever $A, B \in \mathcal{A}$ then $A \subset B$ implies $A = B$.



How large can a chain be? Trivially, a chain $\mathcal{A} \subset \mathcal{P}[n]$ contains at most one element from each $[n]^{(r)}$ so $|\mathcal{A}| \leq n+1$, which is realisable, e.g. $\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \dots, [n]$.

How large can an antichain be? Clearly $[n]^{(r)}$ is an antichain of size $\binom{n}{r}$, which is maximal when $r = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. Can we do better?

Theorem 1.1 (Sperner's Lemma)**.** Let $\mathcal{A} \subset \mathcal{P}[n]$ be an antichain. Then

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. We shall decompose $\mathcal{P}[n]$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains which proves the theorem since $\mathcal{A}$ can meet each chain only once.

It suffices to find injections from $[n]^{(r)}$ to $[n]^{(r+1)}$ so $A \mapsto B$ with $A \subset B$ for $r < \frac{n}{2}$, and from $[n]^{(r)}$ to $[n]^{(r-1)}$ so $A \mapsto B$ with $A \supset B$ for $r > \frac{n}{2}$. The first injection corresponds exactly to a matching of $[n]^{(r)}$ to $[n]^{(r+1)}$ in the bipartite graph whose vertex classes are these two sets, with $A \in [n]^{(r)}$ joined to $B \in [n]^{(r+1)}$ if $A \subset B$.

Observe each $A \in [n]^{(r)}$ is joined to $n-r$ vertices in $[n]^{(r+1)}$. Each $B \in [n]^{(r+1)}$ is joined to $r+1$ vertices in $[n]^{(r)}$. Let $\mathcal{S}$ be a collection of vertices in $[n]^{(r)}$ and $\mathcal{T}$ be its neighbours in $[n]^{(r+1)}$. Counting the edges between $\mathcal{S}$ and $\mathcal{T}$,

$$|\mathcal{S}|(n-r) = e(\mathcal{S}, \mathcal{T}) \leq |\mathcal{T}|(r+1),$$

so

$$|\mathcal{T}| \geq |\mathcal{S}| \frac{n-r}{r+1} \geq |\mathcal{S}|.$$

as $r < \frac{n}{2}$. By Hall's theorem, the matching exists.

The second injection exists similarly. □

Remark. It is not clear from the proof whether the size $\binom{n}{\lfloor n/2 \rfloor}$ can be achieved other than in the obvious ways.

Definition. Given $\mathcal{A} \subset [n]^{(r)}$ the *(lower) shadow* of $\mathcal{A}$ is

$$\partial\mathcal{A} = \partial^{-}\mathcal{A} = \{B \in [n]^{(r-1)} : B \subset A \text{ for some } A \in \mathcal{A}\}.$$

Lemma 1.2 (Local LYM)**.** If $\mathcal{A} \subset [n]^{(r)}$ then

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{\mathcal{A}}{\binom{n}{r}}.$$

Proof. As in the proof of Theorem 1.1,

$$|\mathcal{A}|r = e(\mathcal{A}, \partial\mathcal{A}) \leq |\partial\mathcal{A}|(n - r + 1).$$ □

Remark. (i) Equality is attained only if $\mathcal{A} = \emptyset$ or $\mathcal{A} = [n]^{(r)}$, since we can get from $A \in \mathcal{A}$ to $A' \in [n]^{(r)} \setminus \mathcal{A}$ by a sequence of removing and adding elements.

(ii) We can obtain another proof of Sperner's lemma along these lines. Pick r maximal such that $\mathcal{A} \cap [n]^{(r)} \neq \emptyset$. Replace $\mathcal{A} \cap [n]^{(r)}$ by its shadow if $r > \frac{n}{2}$. By Local LYM, we obtain a larger antichain closer to the middle.

Theorem 1.3 (LYM, Lubell, Yamamoto, Meshalkin, 1966)**.** Let $\mathcal{A} \subset \mathcal{P}[n]$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap [n]^{(r)}|}{\binom{n}{r}} \leq 1.$$

Proof. Let $\mathcal{A}_r = \mathcal{A} \cap [n]^{(r)}$ and

$$\mathcal{B}_r = \mathcal{A}_r \cup \partial\mathcal{A}_{r+1} \cup \cdots \cup \partial^{n-r}\mathcal{A}_n = \mathcal{A}_r \cup \partial\mathcal{B}_{r+1}.$$

Since $\mathcal{A}$ is an antichain, $\mathcal{A}_r \cap \partial\mathcal{B}_{r+1} = \emptyset$. So

$$\begin{aligned} 1 \geq \frac{|\mathcal{B}_0|}{\binom{n}{0}} = \frac{|\mathcal{A}_0|}{\binom{n}{0}} + \frac{|\partial\mathcal{B}_1|}{\binom{n}{0}} \geq \frac{|\mathcal{A}_0|}{\binom{n}{0}} + \frac{|\mathcal{B}_1|}{\binom{n}{1}} = \frac{|\mathcal{A}_0|}{\binom{n}{0}} + \frac{|\mathcal{A}_1|}{\binom{n}{1}} + \frac{|\partial\mathcal{B}_2|}{\binom{n}{1}} \\ \geq \frac{|\mathcal{A}_0|}{\binom{n}{0}} + \frac{|\mathcal{A}_1|}{\binom{n}{1}} + \frac{|\mathcal{B}_2|}{\binom{n}{2}} \geq \cdots \geq \sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}}. \end{aligned}$$ □

Remark. Equality holds in LYM if and only if it holds in Local LYM at every step if and only if $\mathcal{A} = [n]^r$ for some r.

Alternative proof. Pick a random maximal chain $\mathcal{C}$, i.e., a sequence $A_0 \subset A_1 \subset \cdots \subset A_n$ where $|A_r| = r$. Given $A \in [n]^{(r)}$,

$$\mathbb{P}(A \in \mathcal{C}) = \frac{1}{\binom{n}{r}}$$

and hence

$$\mathbb{P}(\mathcal{C} \text{ meets } \mathcal{A}_r) = \frac{|\mathcal{A}_r|}{\binom{n}{r}}.$$

But for $0 \leq r \leq n$ these events are mutually exclusive, so the sum of their probabilities is at most 1. □

Definition. A chain is *symmetric* if it is of the form

$$A_k \subset A_{k+1} \subset \cdots \subset A_{n-k}$$

for some k where $A_i \in [n]^{(i)}$.

Can we decompose $\mathcal{P}[n]$ into symmetric chains? Note there are necessarily $\binom{n}{\lfloor n/2 \rfloor}$ chains, since each chain has an element of $[n]^{(\lfloor n/2 \rfloor)}$.

Theorem 1.4. $\mathcal{P}[n]$ has a partition into symmetric chains.

Proof. By induction on n. Take a partition of $\mathcal{P}[n-1]$ into symmetric chains. Let $\mathcal{B} = A_k, A_{k+1}, \ldots, A_{n-1-k}$ be a chain in it. Let

$$\begin{aligned} \mathcal{B}' &= A_k, A_{k+1}, \ldots, A_{n-1-k}, A_{n-1-k} \cup \{n\} \\ \mathcal{B}'' &= A_k \cup \{n\}, A_{k+1} \cup \{n\}, \ldots, A_{n-2-k} \cup \{n\} \end{aligned}$$

Notice that $\mathcal{B}'$, $\mathcal{B}''$ are symmetric chains in $\mathcal{P}[n]$ and every element of $\mathcal{P}[n]$ is in exactly one of these chains. □

Say what now? We seem to have twice as many chains in $\mathcal{P}[n]$ as in $\mathcal{P}[n-1]$ but $\binom{n}{\lfloor n/2 \rfloor} \neq 2\binom{n-1}{\lfloor (n-1)/2 \rfloor}$ if n is odd. But in this case, $n-1$ is even, $\mathcal{B}$ can have length one, and then $\mathcal{B}'' = \emptyset$. In fact, the procedure generates

$$\ell_k(n) = \binom{n}{k} - \binom{n}{k-1}$$

chains of length $n - 2k + 1$ for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ because $\ell_k(n) = \ell_k(n-1) + \ell_{k-1}(n-1)$.

Littlewood and Offord (1943) needed a bound on the number of sums $\sum_{i \in A} z_i$, $A \subset [n]$, that lie within distance 1 of each other, where $z_1, \ldots, z_n \in \mathbb{C}$, $|z_i| \geq 1$. Erdős (1945) noticed if $z_i \in \mathbb{R}$ then the number is at most $\binom{n}{\lfloor n/2 \rfloor}$, since the sets $A^* = \{i \in A : z_i > 0\} \cup \{i \notin A : z_i < 0\}$ form an antichain.

Theorem 1.5 (Kleitman, 1970)**.** Let $x_1, \ldots, x_n \in X$, where $(X, \|\cdot\|)$ is a normed space, with $\|x_i\| \geq 1$. For any $A \subset [n]$ let $x_A = \sum_{i \in A} x_i$. Let $\mathcal{A} \subset \mathcal{P}[n]$ such that $\|x_A - x_B\| < 1$ for all $A, B \in \mathcal{A}$. Then

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. Call a class $\mathcal{C} \subset \mathcal{P}[n]$ *dispersed* if $\|x_A - x_B\| \geq 1$ for all $A, B \in \mathcal{C}$ with $A \neq B$. If we can partition $\mathcal{P}[n]$ into $\binom{n}{\lfloor n/2 \rfloor}$ dispersed classes, we are done.

A partition of $\mathcal{P}[n]$ into classes is called *quasi-symmetric* if it has $\ell_k(n) = \binom{n}{k} - \binom{n}{k-1}$ classes of size $n - 2k + 1$, $0 \leq k \leq \lfloor n/2 \rfloor$. Notice that any such partition has $\sum_k \ell_k(n) = \binom{n}{\lfloor n/2 \rfloor}$ classes. Notice further that the following procedure produces a quasi-symmetric partition of $\mathcal{P}[n]$ from one of $\mathcal{P}[n-1]$.

For each class $\mathcal{C}$ in the partition of $\mathcal{P}[n-1]$ pick $A^+ \in \mathcal{C}$ and let

$$\begin{aligned} \mathcal{C}' &= \mathcal{C} \cup \{A^+ \cup \{n\}\} \\ \mathcal{C}'' &= \{A \cup \{n\} : A \in \mathcal{C}, A \neq A^+\} \end{aligned}$$

This works because $\ell_k(n) = \ell_k(n-1) + \ell_{k-1}(n-1)$.

We need only pick A^+ so that if $\mathcal{C}$ is dispersed, so are $\mathcal{C}'$ and $\mathcal{C}''$. Notice $\mathcal{C}''$ is dispersed because

$$\|x_{A\cup\{n\}} - x_{B\cup\{n\}}\| = \|x_A - x_B\|$$

for $A, B \in \mathcal{C}''$. Let $e_n = x_n/\|x_n\|$ and pick A^+ so that $\langle x_{A^+}, e_n\rangle = \max_{A\in\mathcal{C}}\langle x_A, e_n\rangle$. Then

$$\begin{aligned}
\|x_{A^+\cup\{n\}} - x_A\| &\geq \langle x_{A^+\cup\{n\}} - x_A, e_n\rangle \\
&= \langle x_{A^+\cup\{n\}}, e_n\rangle - \langle x_A, e_n\rangle \\
&= \langle x_n, e_n\rangle + \langle x_{A^+}, e_n\rangle - \langle x_A, e_n\rangle \\
&\geq \langle x_n, e_n\rangle \\
&= \|x_n\| \\
&\geq 1.
\end{aligned}$$

□

Chapter 2

Saturation

An *r-uniform hypergraph* H is a pair $H = (V, E)$ where $E \subset V^{(r)}$. A 2-uniform hypergraph is a graph. The *complete hypergraph* $K_k^{(r)}$ of order k is $([k], [k]^{(r)})$.

Recall that for $r = 2$, Turán's theorem tells us the maximum size, i.e., the number of edges, of a graph with no K_k is that of the Turán graph $T_{k-1}(n)$. The corresponding value for $r \geq 3$ is completely unknown, even for $K_4^{(3)}$.

A hypergraph is *(strongly) k-saturated* if the addition of any edge not in H produces a $K_k^{(r)}$ subgraph. For $r = 2$, clearly $T_{k-1}(n)$ is saturated, but there are examples with fewer edges. E.g., for $k = 3$ a star will do and in general $K_{k-2} + E_{n-k+2}$ works. Erdős–Hajnal–Moon (1964) showed this example has the minimum size.

Theorem 2.1 (Bollobás, 1965)**.** Let $\{(R_i, S_i) : i \in I\}$ be a collection of pairs of subsets of $[n]$ such that $R_i \cap S_j = \emptyset$ if and only if $i = j$. Then

$$\sum_{i \in I} \binom{r_i + s_i}{r_i}^{-1} \leq 1$$

where $r_i = |R_i|$, $s_i = |S_i|$.

Remark. Putting $S_i = [n] - R_i$, $\{R_i : i \in I\}$ forms an antichain and we obtain the LYM inequality.

Proof. By induction on n. The case $n = 1$ is trivial. For each $x \in [n]$, let $I_x = \{i \in I : x \notin R_i\}$. Let $S_j^x = S_j \cap ([n] - \{x\})$. Then by the induction hypothesis,

$$\sum_{i \in I_x} \binom{r_i + s_i^x}{r_i}^{-1} \leq 1.$$

Now R_i appears in $n - r_i$ I_x's and, of these, $s_i^x = s_i - 1$ in s_i cases, and $s_i^x = s_i$ in $n - r_i - s_i$ cases. So

$$\begin{aligned} n &\geq \sum_{x \in [n]} \sum_{i \in I_x} \binom{r_i + s_i^x}{r_i}^{-1} \\ &= \sum_{i \in I} (n - r_i - s_i) \binom{r_i + s_i}{r_i}^{-1} + s_i \binom{r_i + s_i - 1}{r_i}^{-1} \\ &= n \sum_{i \in I} \binom{r_i + s_i}{r_i}^{-1}. \end{aligned}$$

□

Proof. Consider a random ordering of $[n]$, i.e., a random maximal chain. Then

$$\mathbb{P}(\text{All elements of } R_i \text{ precede all elements of } S_i) = \frac{1}{\binom{r_i+s_i}{r_i}}$$

and those events are mutually disjoint. □

Theorem 2.2. Let H be a strongly $(r+t)$-saturated r-uniform hypergraph. Then

$$e(H) \geq \binom{n}{r} - \binom{n-t}{r}$$

where $n = |H|$.

Remark. This is attained (in fact uniquely) by $[n]^{(r)} - [n-t]^{(r)}$.

Proof. For each missing edge R_i pick a $K^{(r)}_{r+t}$ created by the addition of R_i and let $S_i = [n] - V(K^{(r)}_{r+t})$. Then $R_i \cap S_j = \emptyset$ if and only if $i = j$ so the number of R_i is at most $\binom{n-t}{r}$. □

We say that H is *weakly k-saturated* if there is an ordering of the missing edges so that if the edges are added one by one in that order, each additional edge creates a new $K^{(r)}_k$. A strongly saturated graph is weakly saturated. The converse is false, e.g., if $r = 2$, $k = 3$ then any tree will do.

Theorem 2.3. Let H be a weakly $(r+t)$-saturated r-uniform hypergraph. Then

$$e(H) \geq \binom{n}{r} - \binom{n-t}{r}$$

where $n = |H|$.

Proof. For each missing edge R_i pick S_i as before. We add edges in the order $R_1, R_2, \ldots$. Then $R_i \cap S_i = \emptyset$ and $R_j \cap S_i \neq \emptyset$ for $j > i$. The proof then follows from the Theorem 2.4. □

Theorem 2.4. Let $R_i, S_i \subset [n]$, $1 \leq i \leq h$ with $|R_i| = r$, $|S_i| = s$, $R_i \cap S_i = \emptyset$, and $R_j \cap S_i \neq \emptyset$ for $1 \leq i < j \leq h$. Then $h \leq \binom{r+s}{r}$.

Remark. Given a finite dimensional vector space V with basis $e_1, \ldots, e_d$, we consider the exterior algebra $\bigwedge = \bigoplus_{k\geq 1} \bigwedge^k V$. A basis for $\bigwedge^k V$ is

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

where $\{i_1, \ldots, i_k\} \in [d]^{(k)}$ and we require that transposing elements results in multiplication by -1; so if any two elements are equal we obtain zero. Extend this by linearity. It is easy to check that $v_1 \wedge \cdots \wedge v_k \neq 0$ iff $\{v_1, \ldots, v_k\}$ is a linearly independent set.

Proof. Let $V = \mathbb{R}^{r+s}$, let $\{v_x : x \in [n]\}$ be vectors in general position, that is, any $r+s$ are linearly independent. For $A \subset [n]$ let $v_A = \bigwedge_{x\in A} v_x$. Then $v_A \wedge v_B \neq 0$ if and only if $A \cap B = \emptyset$ and $\{v_x : x \in A \cup B\}$ is linearly independent.

Now $v_{R_i} \wedge v_{S_i} \neq 0$, but $v_{R_j} \wedge v_{S_i} = 0$ for $i < j$. Then $\{v_{R_i} : 1 \leq i \leq h\}$ is linearly independent, for if $\sum_{j=1}^{k} c_j v_{R_j} = 0$ let $i = \min\{j : c_j \neq 0\}$ then

$$\left(\sum_{j=1}^{k} c_j v_{R_j}\right) \wedge v_{S_i} = c_i v_{R_i} \wedge v_{S_i} = 0$$

so $c_i = 0$, contradiction. Therefore,

$$h \leq \dim \bigwedge^{r} V = \binom{\dim V}{r} = \binom{r+s}{r} \qquad \square$$

Chapter 3

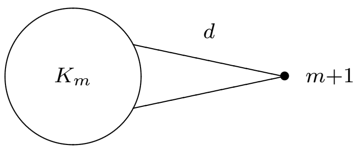
Shadows

Local LYM states

$$|\partial\mathcal{A}| \geq |\mathcal{A}|\frac{r}{n-r+1}$$

with equality only if $\mathcal{A} = \emptyset$ or $\mathcal{A} = [n]^{(r)}$. How small can $|\partial\mathcal{A}|$ be, given $|\mathcal{A}|$?

If $r = 1$, clearly $\partial\mathcal{A} = \{\emptyset\}$, $|\partial\mathcal{A}| = 1$. If $r = 2$, $\mathcal{A}$ represents the edges of a graph. $\partial\mathcal{A}$ is the set of vertices having at least one incident edge. We minimise $|\partial\mathcal{A}|$ by



where $|\mathcal{A}| = \binom{m}{2} + d$, $0 < d \leq m$. Hence $|\partial\mathcal{A}| \geq m + 1$. If $d \neq m$ there are other configurations.

The following are two important orders on $[n]^{(r)}$. We define the *lexicographic* (or *lex*) order by $A \prec B$ if $\min A \triangle B \in A$, and the *colexicographic* (or *colex*) order by $A \prec B$ if $\max A \triangle B \in B$. For example, consider $[5]^{(3)}$.

lex	123	124	125	134	135	145	234	235	245	345
colex	123	124	134	234	125	135	235	145	245	345

Remark. Colex is “lex reversed on $[n]$ reversed”.

Note if $\mathcal{A} \subset [n]^{(2)}$ then we minimised the shadow by choosing $\mathcal{A}$ to be an initial segment of the colex order.

Clearly, in colex, every set with maximal element less than k precedes any set with maximal element k. Amongst sets with maximal element k, the order is by colex on $[k-1]^{(r-1)} + \{k\}$.

Example. The first 41 elements in $[n]^{(4)}$.

$$41 = \binom{7}{4} + \binom{4}{3} + \binom{2}{2} + \binom{1}{1}$$

$$\mathcal{A} = [7]^{(4)} \cup \Big([4]^{(3)} + \{8\}\Big) \cup \Big([2]^{(2)} + \{5,8\}\Big) \cup \Big([1]^{(1)} + \{3,5,8\}\Big)$$

$$\partial\mathcal{A} = [7]^{(3)} \cup \Big([4]^{(2)} + \{8\}\Big) \cup \Big([2]^{(1)} + \{5,8\}\Big) \cup \Big([1]^{(0)} + \{3,5,8\}\Big)$$

$$|\partial\mathcal{A}| = \binom{7}{3} + \binom{4}{2} + \binom{2}{1} + \binom{1}{0} = 44$$

Observation 3.1. Each $m \in \mathbb{N}$ has, given fixed r, a unique expression

$$m = \binom{m_r}{r} + \binom{m_{r-1}}{r-1} + \cdots + \binom{m_s}{s}$$

where $m_r > m_{r-1} > \cdots > m_s \geq s \geq 1$. If $\mathcal{I}$ is the initial segment of length m in the colex order on $[n]^{(r)}$ then

$$|\partial \mathcal{I}| = \binom{m_r}{r-1} + \binom{m_{r-1}}{r-2} + \cdots + \binom{m_s}{s-1}.$$

Observation 3.2. Let $1 \in A \in [n]^{(r)}$ and let B be the first element after A in colex with $1 \notin B$. Then $A - \{1\} \subset B$.

To check this, consider the maximal string of consecutive numbers in A beginning with 1, i.e., write $A = [l] \cup C$, $l+1 \notin A$. The next elements in colex are $\{1, 2, \ldots, l-1, l+1\} \cup C$, $\{1, 2, \ldots, l-2, l, l+1\} \cup C$, $\ldots$, $\{1, 3, 4, \ldots, l+1\} \cup C$, $\{2, 3, \ldots, l+1\} \cup C = B$.

Our hope is to transform a family $\mathcal{A}$ to another family more like an initial segment, *without* increasing the shadow. The last bit is the hard bit.

Definition (i-j-compression)**.** Let $1 \leq i < j \leq n$, $A \in [n]^{(r)}$ then

$$C_{ij}(A) = \begin{cases} A - \{j\} \cup \{i\} & \text{if } j \in A,\ i \notin A \\ A & \text{otherwise} \end{cases}.$$

For $\mathcal{A} \subset [n]^{(r)}$ its i-j-compression is

$$C_{ij}\mathcal{A} = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}(A) \in \mathcal{A}\}$$

replacing sets A by their compression if possible.

Example.

$$C_{12}\{125, 146, 156, 256, 257, 357\} = \{125, 146, 156, 256, 157, 357\}$$

Clearly, $|C_{ij}\mathcal{A}| = |\mathcal{A}|$.

Lemma 3.3.

$$|\partial C_{ij}\mathcal{A}| \leq |\partial \mathcal{A}|$$

Proof. We consider the injection $\mathcal{P}[n] \to \mathcal{P}[n], B \mapsto B \triangle \{i, j\}$ and show it maps $\partial C_{ij}\mathcal{A} \setminus \partial \mathcal{A}$ to $\partial \mathcal{A} \setminus \partial C_{ij}\mathcal{A}$, proving the lemma.

Let $B \in \partial C_{ij}\mathcal{A} \setminus \partial \mathcal{A}$. Then there exists $A \in \mathcal{A}$ with $B \subset C_{ij}(A)$ with $C_{ij}(A) \notin \mathcal{A}$ so $C_{ij}(A) \neq A$. Then $A = Z \cup \{j\}$, $C_{ij}(A) = Z \cup \{i\}$, $Z \in [n]^{(r-1)}$, $i, j \notin Z$.

Now $B \neq Z$ since $B \notin \partial \mathcal{A}$, so $B = Z - \{k\} \cup \{i\}$, $k \in Z$. So $B \mapsto B' = Z - \{k\} \cup \{j\} \subset A$, so $B' \in \partial \mathcal{A}$.

Suppose $B' \in \partial C_{ij}\mathcal{A}$, i.e., $B' \subset A' \in C_{ij}\mathcal{A}$. Then $A' = B' \cup \{l\} = Z - \{k\} \cup \{j\} \cup \{l\}$. If $l = i$ then $A' \in \mathcal{A}$ and $B \subset A'$ so $B \in \partial \mathcal{A}$, contradiction. But if $l \neq i$ then $A' = Z - \{k\} \cup \{j\} \cup \{l\}$ can be in $C_{ij}\mathcal{A}$ only if $C_{ij}(A') = Z - \{k\} \cup \{i\} \cup \{l\} \in \mathcal{A}$. So $B \subset C_{ij}A'$, i.e., $B \in \partial \mathcal{A}$, contradiction.

Hence $B' \notin \partial C_{ij}\mathcal{A}$, completing the proof. ∎

We say $\mathcal{A}$ is *i-j-compressed* if $C_{ij}\mathcal{A} = \mathcal{A}$. We say $\mathcal{A}$ is *left-compressed* if $C_{ij}\mathcal{A} = \mathcal{A}$ for all $i < j$.

Corollary 3.4. Given $\mathcal{A} \subset [n]^{(r)}$ there exist $\mathcal{B} \subset [n]^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$, $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$ and $\mathcal{B}$ is left-compressed.

Proof. If $C_{ij}\mathcal{A} \neq \mathcal{A}$ replace $\mathcal{A}$ by $C_{ij}\mathcal{A}$. This reduces $\sum_{A\in\mathcal{A}}\sum_{i\in A} i$. Keep doing this; since the quantity is positive we reach a left-compressed family $\mathcal{B}$, which has the desired properties by Lemma 3.3. □

An initial segment of colex is left-compressed but unfortunately there are many more other examples, e.g., an initial segment of lex. Nevertheless, we have enough for our main theorem.

Theorem 3.5 (Kruskal–Katona, 1963–1968)**.** Let $\mathcal{A} \subset [n]^{(r)}$ and let $\mathcal{J}$ be the initial segment of colex on $[n]^{(r)}$ with $|\mathcal{A}| = |\mathcal{J}|$. Then $|\partial\mathcal{A}| \geq |\partial\mathcal{J}|$. Explicitly, if

$$|\mathcal{A}| = m = \binom{m_r}{r} + \binom{m_{r-1}}{r-1} + \cdots + \binom{m_s}{s}$$

where $m_r > m_{r-1} > \cdots > m_s \geq s \geq 1$, then

$$|\partial\mathcal{A}| \geq \binom{m_r}{r-1} + \cdots + \binom{m_s}{s-1}.$$

Proof. Proceed by induction on $m + r$. By Corollary 3.4, we may assume $\mathcal{A}$ is 1-j-compressed for all $1 < j$. Let us define

$$\mathcal{A}_1 = \{A - \{1\} : 1 \in A \in \mathcal{A}\} \qquad \mathcal{J}_1 = \{A - \{1\} : 1 \in A \in \mathcal{J}\}$$
$$\mathcal{A}_2 = \{A \in \mathcal{A} : 1 \notin A\} \qquad \mathcal{J}_2 = \{A \in \mathcal{J} : 1 \notin A\}$$

Notice that $\mathcal{J}_1$ is an initial segment of colex on $\{2, \ldots, n\}^{(r-1)}$ and $\mathcal{J}_2$ is an initial segment of colex on $\{2, \ldots, n\}^{(r)}$. Then $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|$ and $|\mathcal{J}| = |\mathcal{J}_1| + |\mathcal{J}_2|$.

Now $\mathcal{A}$ is 1-j-compressed means $\partial\mathcal{A}_2 \subset \mathcal{A}_1$. Thus $\partial\mathcal{A} = \mathcal{A}_1 \cup (\{1\} + \partial\mathcal{A}_1)$ is a partition of $\partial\mathcal{A}$. So $|\partial\mathcal{A}| = |\mathcal{A}_1| + |\partial\mathcal{A}_1|$, and likewise $|\partial\mathcal{J}| = |\mathcal{J}_1| + |\partial\mathcal{J}_1|$.

So we are home by induction if $|\mathcal{A}_1| \geq |\mathcal{J}_1|$. What if $|\mathcal{A}_1| < |\mathcal{J}_1|$? Then $|\mathcal{A}_2| > |\mathcal{J}_2|$. Let $\mathcal{J}_2^+$ be $\mathcal{J}_2$ plus the next element in colex on $\{2, \ldots, n\}^{(r)}$. Then $\mathcal{J}_2^+$ is an initial segment of colex and $|\mathcal{A}_2| \geq |\mathcal{J}_2^+|$. Then $\mathcal{J}_1 \subset \partial\mathcal{J}_2^+$ by Observation 3.2.

But, recalling $\partial\mathcal{A}_2 \subset \mathcal{A}_1$, we now obtain $|\mathcal{A}_1| \geq |\partial\mathcal{A}_2| \geq |\partial\mathcal{J}_2^+| \geq |\mathcal{J}_1|$ using the induction hypothesis, contradiction. □

Theorem 3.6. Let $\mathcal{A} \subset [n]^{(r)}$ with $|\mathcal{A}| = \binom{x}{r}$ where $x > r - 1$ then

$$|\partial\mathcal{A}| \geq \binom{x}{r-1}.$$

Proof (Lovász 1979). By Theorem 3.5. □

Proof (Frankl 1984). Since $\binom{x}{r-1}$ is increasing for $x > r - 1$ and $|\mathcal{A}| \geq 1$ we have $x \geq r$. Moreover, if $x = r$ the result is trivial so we may assume $x > r$.

Proceed as in the proof of Theorem 3.5; $\mathcal{A}$ is 1-j-compressed, $\partial\mathcal{A}_2 \subset \mathcal{A}_1$ and $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|$, $|\partial\mathcal{A}| = |\mathcal{A}_1| + |\partial\mathcal{A}_1|$. If $|\mathcal{A}_1| \geq \binom{x-1}{r-1}$ then $|\partial\mathcal{A}_1| \geq \binom{x-1}{r-2}$ by induction so $|\partial\mathcal{A}| \geq \binom{x}{r-1}$. If $|\mathcal{A}_1| < \binom{x-1}{r-1}$ then $|\mathcal{A}_2| > \binom{x-1}{r}$ and since $x > r$, induction shows $|\partial\mathcal{A}_2| \geq \binom{x-1}{r-1} > |\mathcal{A}_1|$, contradicting $\partial\mathcal{A}_2 \subset \mathcal{A}_1$. □

An initial segment of lex is left-compressed. But we could move 125 to 234 using $C_{34,15}$. Given $U, V \subset [n]$, $U \cap V = \emptyset$ and $A \subset [n]$ we define

$$C_{UV}(A) = \begin{cases} (A \setminus V) \cup U & \text{if } V \subset A \text{ and } U \cap A = \emptyset \\ A & \text{otherwise} \end{cases}$$

and

$$C_{UV}\mathcal{A} = \{C_{UV}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{UV}(A) \in \mathcal{A}\}$$

Note $C_{\{i\}\{j\}} = C_{ij}$ and $|C_{UV}\mathcal{A}| = |\mathcal{A}|$. If $\mathcal{A} \subset [n]^{(r)}$ and $|U| = |V|$ then $C_{UV}\mathcal{A} \subset [n]^{(r)}$. But note that

$$\mathcal{A} = \{14, 15\} \qquad \partial\mathcal{A} = \{1, 4, 5\}$$
$$C_{23,14}\mathcal{A} = \{23, 15\} \qquad \partial C_{23,14}\mathcal{A} = \{1, 2, 3, 5\}$$

However, we have the following lemma.

Lemma 3.7 (Bollobás–Leader, 1987)**.** Let $\mathcal{A} \subset [n]^{(r)}$, $U \cap V = \emptyset$, $|U| = |V|$. Suppose

$$\forall u \in U \quad \exists v \in V \quad C_{U-\{u\},V-\{v\}}\mathcal{A} = \mathcal{A} \tag{†}$$

Then $|\partial C_{UV}\mathcal{A}| \leq |\partial\mathcal{A}|$.

Proof. We show that the bijection of $\mathcal{P}[n] \to \mathcal{P}[n]$ given by $Y \mapsto Y \triangle (U \cup V)$ injects $\partial\mathcal{A}' \setminus \partial\mathcal{A}$ into $\partial\mathcal{A} \setminus \partial\mathcal{A}'$ where $\cdot'$ denotes $C_{UV}\cdot$.

Let $B \in \partial\mathcal{A}' \setminus \partial\mathcal{A}$. So there exists $x \in [n]$ such that $B \cup \{x\} \in \mathcal{A}' \setminus \mathcal{A}$. Thus $U \subset B \cup \{x\}$ and $V \cap (B \cup \{x\}) = \emptyset$. Thus $(B \cup \{x\} \setminus U) \cup V \in \mathcal{A}$. Now $x \notin U$, else by (†) there exists $v \in V$ with $C_{U-\{u\},V-\{v\}}(B \cup \{x\} \setminus U) \cup V = B \cup \{v\} \in \mathcal{A}$, implying $B \in \partial\mathcal{A}$. Thus $x \notin U \cup V$, so $B \triangle (U \cup V) = (B \setminus U) \cup V \in \partial\mathcal{A}$. Suppose that $(B \setminus U) \cup V \in \partial\mathcal{A}'$. Then there exists y such that $(B \setminus U) \cup V \cup \{y\} \in \mathcal{A}'$. Suppose $y \notin U$. Then $B \cup \{y\} = C_{UV}(B \setminus U) \cup V \cup \{y\} \in \mathcal{A}$, giving $B \in \partial\mathcal{A}$, a contradiction. Hence $y \in U$. Then by (†) there exists $v \in V$ such that $C_{U-\{u\},V-\{v\}}\mathcal{A} = \mathcal{A}$. So both $(B \setminus U) \cup V \cup \{y\} \in \mathcal{A}'$ and $C_{U-\{u\},V-\{v\}}(B \setminus U) \cup V \cup \{y\} = B \cup \{v\} \in \mathcal{A}$, so $B \in \partial\mathcal{A}$.

Thus $B \triangle (U \cup V) = (B \setminus U) \cup V \in \partial\mathcal{A} \setminus \partial\mathcal{A}'$ as claimed. □

Definition.

$$\Gamma = \{(U, V) \in \mathcal{P}[n] \times \mathcal{P}[n] : U \cap V = \emptyset, |U| = |V|, \max U < \max V\}$$

Lemma 3.8. $\mathcal{A}$ is an initial segment of colex if and only if $C_{UV}\mathcal{A} = \mathcal{A}$ for all $(U, V) \in \Gamma$.

Proof. If $\mathcal{A}$ is not an initial segment pick $A' \notin \mathcal{A}$, $A \in \mathcal{A}$ where $A' \prec A$. Let $U = A' \setminus A$, $V = A \setminus A'$. Then $C_{UV}(A) = A'$, so $C_{UV}\mathcal{A} \neq \mathcal{A}$, and $\max(U \cup V) = \max(A \triangle A') \in A \subset V$, i.e., $(U, V) \in \Gamma$.

If $C_{UV}\mathcal{A} \neq \mathcal{A}$ pick $A \in \mathcal{A}$ with $C_{UV}A \notin \mathcal{A}$, then $\max(C_{UV}(A) \triangle A) = \max(U \cup V) \in A$, so $C_{UV}(A) \prec A$ and $\mathcal{A}$ is not an initial segment. □

Proof (Kruskal–Katona). Given $\mathcal{A}$ which is not an initial segment of colex, pick $(U, V) \in \Gamma$ with $C_{UV}\mathcal{A} \neq \mathcal{A}$ and $|U|$ minimal, by Lemma 3.8. By Lemma 3.7,

$$|\partial C_{UV}\mathcal{A}| \leq |\partial \mathcal{A}|.$$

Repeat; since the members of $C_{UV}\mathcal{A}$ are to the left of those in $\mathcal{A}$, we cannot repeat forever. □

What about minimising $|\partial^+\mathcal{A}|$? This depends on n.

Corollary 3.9. Let $\mathcal{A} \subset [n]^{(r)}$ and let $\mathcal{J}$ be the initial segment of $[n]^{(r)}$ in the lex order with $|\mathcal{A}| = |\mathcal{J}|$. Then $|\partial^+\mathcal{A}| \geq |\partial^+\mathcal{J}|$.

Proof. Define $\bar{A} = \{[n] - A : A \in \mathcal{A}\} \subset [n]^{(n-r)}$. Then $\partial^+\mathcal{A} = \overline{\partial^-\bar{A}}$; we use Kruskal–Katona and the relationship between lex and colex. □

Chapter 4

Intersecting Systems

We say $\mathcal{A} \subset \mathcal{P}[n]$ is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. For example, $\mathcal{A} = \{A \subset [n] : 1 \in A\}$ is intersecting and $|\mathcal{A}| = 2^{n-1}$.

Proposition 4.1. If $\mathcal{A} \subset \mathcal{P}[n]$ is intersecting then $|\mathcal{A}| \leq 2^{n-1}$.

Proof. $\mathcal{A}$ can contain at most one of each pair $A, [n] - A$. □

What about uniform intersecting systems? Note if $r > \frac{n}{2}$ then $[n]^{(r)}$ is intersecting. If $r = \frac{n}{2}$ then any choice of one from each pair $A, [n] - A$ gives an intersecting family of size $\frac{1}{2}\binom{n}{r} = \binom{n-1}{r-1}$. If $r < \frac{n}{2}$ the family $\{A \in [n]^{(r)} : 1 \in A\}$ is intersecting and has size $\binom{n-1}{r-1}$.

Theorem 4.2 (Erdős–Ko–Rado, 1938, 1961)**.** Let $\mathcal{A} \subset [n]^{(r)}$ with $r \leq \frac{n}{2}$ be intersecting. Then

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

Proof. Let $\bar{\mathcal{A}} = \{[n] - A : A \in \mathcal{A}\} \subset [n]^{(n-r)}$. The fact that $\mathcal{A}$ is intersecting is precisely the statement that $\mathcal{A} \cap \partial^{n-2r}\bar{\mathcal{A}} = \emptyset$. If

$$|\mathcal{A}| > \binom{n-1}{r-1}$$

then

$$|\bar{\mathcal{A}}| = |\mathcal{A}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$$

so, by Kruskal–Katona $n - 2r$ times,

$$|\partial^{n-2r}\bar{\mathcal{A}}| \geq \binom{n-1}{r}.$$

But then

$$|\mathcal{A}| + |\partial^{n-2r}\bar{\mathcal{A}}| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r},$$

contradiction. □

Proof (Katona). Consider all $n!$ cyclic orders of the n elements. A set $A \in \mathcal{A}$ appears as an arc in $n \cdot r! \cdot (n-r)!$ orders. Given a fixed cyclic ordering at most r arcs can represent sets in $\mathcal{A}$. (If $(c_1, \ldots, c_r)$ is an arc in $\mathcal{A}$ then for $1 \le i \le r-1$ at most one of the arcs $(\cdot, \cdot, \ldots, c_i)$ and $(c_{i+1}, \cdot, \cdot, \ldots)$ is in $\mathcal{A}$.) Thus

$$|\mathcal{A}| \cdot n \cdot r! \cdot (n-r)! \le r \cdot n!. \qquad \square$$

We say $\mathcal{A}$ is *t-intersecting* if $|A \cap B| \ge t$ for $A, B \in \mathcal{A}$. Clearly, intersecting means 1-intersecting. We consider the non-uniform case first.

Lemma 4.3. Let $\mathcal{A} \subset \mathcal{P}[n]$ be t-intersecting. Then $C_{UV}\mathcal{A}$ is t-intersecting provided $|U| \ge |V|$ and

(i) $C_{U,V-\{v\}}\mathcal{A} = \mathcal{A}$ for all $v \in V$,
(ii) for all $u \in U$ there exists $v \in V$ such that $C_{U-\{u\},V-\{v\}}\mathcal{A} = \mathcal{A}$.

Proof. Suppose not. Then there exist $A, B \in C_{UV}\mathcal{A}$ with $|A \cap B| < t$. Clearly not both $A, B \in \mathcal{A}$, so we assume $A = C_{UV}A'$, $A' \in \mathcal{A}$, $A \notin \mathcal{A}$ so $A = (A' - V) \cup U$.

Suppose $B \notin \mathcal{A}$. Then $B = (B' - V) \cup U$ where $B' \in \mathcal{A}$. Then

$$|A \cap B| = |(A' - V \cup U) \cap (B' - V \cup U)| = |A' \cap B'| + |U| - |V| \ge t$$

contradiction. So $B \in \mathcal{A}$. Suppose $C_{UV}B \ne B$. Then $C_{UV}B \in \mathcal{A}$ since $B \in C_{UV}\mathcal{A}$. So

$$|A \cap B| = |(A' - V \cup U) \cap B| = |A' \cap (B - V \cup U)| = |A' \cap C_{UV}B| \ge t$$

contradiction. So $C_{UV}B = B$. Hence either $V \not\subset B$ or $V \subset B$ but $U \cap B \ne \emptyset$. If $V \not\subset B$ take $v \in V$, $v \notin B$. By (i) $C_{U,V-\{v\}}A' \in \mathcal{A}$. Then

$$|A \cap B| = |(A' - V \cup U) \cap B| = |(A' - V \cup \{v\} \cup U) \cap B| = |C_{U,V-\{v\}}A' \cap B| \ge t$$

contradiction. Finally, if $V \subset B$ but $U \cap B \ne \emptyset$, take $u \in U \cap B$ and by (ii) take $v \in V$ so that $C_{U-\{u\},V-\{v\}}\mathcal{A} = \mathcal{A}$. Then

$$|A \cap B| = |(A' - (V - \{v\}) \cup (U - \{u\})) \cap B| = |C_{U-\{u\},V-\{v\}}A' \cap B| \ge t$$

since $C_{U-\{u\},V-\{v\}}A' \in \mathcal{A}$. $\square$

Theorem 4.4 (Katona, 1964)**.** Let $\mathcal{A} \subset \mathcal{P}[n]$ be t-intersecting. Then

$$|\mathcal{A}| \le \left|[n]^{(\ge k)}\right| = \sum_{i=k}^{n} \binom{n}{i}$$

if $n + t = 2k$, and

$$|\mathcal{A}| \le \left|[n]^{(>k)} \cup [n-1]^{(k)}\right| = \binom{n-1}{k} + \sum_{i=k+1}^{n} \binom{n}{i}$$

if $n + t = 2k + 1$.

Proof. Consider all pairs (U, V) with $|U| > |V|$, $U \cap V = \emptyset$. Keep choosing such a pair with $|V|$ minimal and $C_{UV}\mathcal{A} \neq \mathcal{A}$, if such a pair exists. If $V = \emptyset$, $C_{UV}\mathcal{A}$ is trivially t-intersecting, and if $V \neq \emptyset$ then $C_{UV}\mathcal{A}$ is t-intersecting by Lemma 4.3 and the minimality of $|V|$. Replace $\mathcal{A}$ by $C_{UV}\mathcal{A}$. This increases $\sum_{A\in\mathcal{A}}|A|$ so eventually we reach a family with $C_{UV}\mathcal{A} = \mathcal{A}$ for all pairs (U, V).

Define $r = \min\{|A| : A \in \mathcal{A}\}$. Then $[n]^{(j)} \subset \mathcal{A}$ for all $j > r$: else let $A \in \mathcal{A} \cap [n]^{(r)}$ and $B \in [n]^{(j)} \setminus \mathcal{A}$, put $U = B - A$, $V = A - B$, and note $C_{UV}\mathcal{A} \neq \mathcal{A}$.

Now pick $A \in \mathcal{A} \cap [n]^{(r)}$ and $B \in [n]^{(r+1)}$ with $|A \cap B| = r + (r + 1) - n$. Since $B \in \mathcal{A}$, we have $2r + 1 - n \geq t$, so $r \geq k$.

If $n + t$ is even, we are done; because $\mathcal{A} \subset [n]^{(\geq k)}$ and the latter is t-intersecting. If $n + t$ is odd, $\mathcal{A} \subset [n]^{(>k)} \cup (\mathcal{A} \cap [n]^{(k)})$ which is t-intersecting if and only if $\mathcal{A} \cap [n]^{(k)}$ is t-intersecting. This condition is equivalent to $|A \cup B| < n$ if $A, B \in \mathcal{A} \cap [n]^{(k)}$, which is equivalent to $\{[n] - A : A \in \mathcal{A} \cap [n]^{(k)}\}$ is an intersecting $(n - k)$-uniform family. Since $n - k \leq \frac{n}{2}$, Erdős–Ko–Rado says

$$|\mathcal{A} \cap [n]^{(k)}| \leq \binom{n-1}{n-k-1} = \binom{n-1}{k}. \qquad \square$$

Theorem 4.5. Let $1 \leq t \leq r$ and let $\mathcal{A} \subset [n]^{(r)}$ be t-intersecting. If n is sufficiently large, e.g., $n \geq (16r)^r$, then

$$|\mathcal{A}| \leq \binom{n-t}{r-t}.$$

Proof. We may assume $t < r$ and $\mathcal{A}$ is maximal t-intersecting. Then we may choose $A, B \in \mathcal{A}$ with $|A \cap B| = t$.

If $Y \supset A \cap B$ for all $Y \in \mathcal{A}$ then

$$|\mathcal{A}| \leq \binom{n-t}{r-t}$$

So suppose there exists $C \in \mathcal{A}$ with $A \cap B \not\subset C$. Thus, if $D \in \mathcal{A}$ then $|D \cap (A \cup B \cup C)| \geq t + 1$. Thus

$$\begin{aligned}|\mathcal{A}| &\leq 2^{|A\cup B\cup C|}\left[\binom{n}{r-t-1} + \binom{n}{r-t-2} + \cdots + \binom{n}{0}\right]\\ &< \binom{n-t}{r-t}\end{aligned}$$

if n is large. $\square$

Theorem 4.5 fails if n is not large. Let

$$\mathcal{F}_i = \{A \in [n]^{(r)} : |A \cap [t + 2i]| \geq t + i\}.$$

These are t-intersecting families, interpolating between two configurations.

Example. Let $r = 4$ and $t = 2$.

n	$\lvert\mathcal{F}_0\rvert$	$\lvert\mathcal{F}_1\rvert$	$\lvert\mathcal{F}_2\rvert$
7	$\binom{5}{2} = 10$	$1 + \binom{4}{3}\binom{3}{1} = 13$	$\binom{6}{4} = 15$
8	$\binom{6}{2} = 15$	$1 + \binom{4}{3}\binom{4}{1} = 17$	$\binom{6}{4} = 15$
9	$\binom{7}{2} = 21$	$1 + \binom{4}{3}\binom{5}{1} = 21$	$\binom{6}{4} = 15$

Frankl (1987) conjectured one of $\mathcal{F}_i$ always wins. In particular, $\mathcal{F}_0$ is biggest if $n > (r-t+1)(t+1)$, which was proved by Wilson in 1984 (c.f. Theorem 4.5). The conjecture was proved by Ahlswede and Khachatrian in 1997.

There remain many beautiful open problems. These two are both due to Simonovits and Sós.

(i) If $\mathcal{A} \subset \mathcal{P}[n]$ such that $|A \cap B|$ contains a 3-term arithmetic progression, then $|\mathcal{A}| \leq 2^{n-3}$.

(ii) If $\mathcal{A} \subset \mathcal{P}[n]$ is a family of graphs on the vertex set $[n]$ such that $|A \cap B|$ contains a triangle then $|\mathcal{A}| \leq 2^{\binom{n}{2}-3}$. It is known that $|\mathcal{A}| \leq 2^{\binom{n}{2}-2}$.

Chapter 5

Exact Intersections

Historical notes from statistics: a (r, k)-λ *design* is a family $\mathcal{A} \subset [v]^{(r)}$ for some v, whose members are called blocks, every element of $[v]$ lies in exactly k blocks, and every pair of elements of $[v]$ lies in λ blocks.

Clearly, the parameters are constrained. For example, if $b = |\mathcal{A}|$ then $br = vk$ and $\lambda\binom{v}{2} = b\binom{r}{2}$. A less apparent constraint is $b \geq v$, called *Fisher's Inequality.* It turns out to hold more generally: we need only $\mathcal{A} \subset \mathcal{P}[v]$ with every pair in $[v]$ lying in λ blocks.

The *dual system* to $\mathcal{A}$ is $\mathcal{A}^* = \{\mathcal{A}_x : x \in [v]\}$ where $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$. Think of a bipartite graph with vertex class $[v]$ and $\mathcal{A}$ with edges representing containment. The dual version of Fisher is the one we shall prove.

Theorem 5.1 (Fisher's Inequality)**.** Let $\mathcal{A} \subset \mathcal{P}[n]$ and let $\lambda \in \mathbb{N} \cup \{0\}$ be such that $|A \cap B| = \lambda$ for distinct $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq n$, unless $\lambda = 0$ then $|\mathcal{A}| \leq n + 1$.

Proof. If $|A| = \lambda$ for some $A \in \mathcal{A}$ then $B \supset A$ for all $B \in \mathcal{A}$ and the sets $B \setminus A$ are pairwise disjoint, so $|\mathcal{A}| \leq 1 + n - |A|$.

So we may assume $|A| > \lambda$ for all $A \in \mathcal{A}$. For $A \in \mathcal{A}$ let $x_A \in \mathbb{R}^n$ be its characteristic vector, i.e. $x_A = (\delta_1, \ldots, \delta_n)$ where $\delta_i = 1$ if $i \in A$ and $\delta_i = 0$ if $i \notin A$. Then $x_A \cdot x_A = |A|$, and $x_A \cdot x_B = |A \cap B| = \lambda$ if $A \neq B$.

Suppose now $\sum_{A \in \mathcal{A}} c_A x_A = 0$ where $c_A \in \mathbb{R}$. Then dotting with x_B we obtain

$$c_B(|B| - \lambda) = -\lambda C$$

for $C = \sum_{A \in \mathcal{A}} c_A$. If $\lambda = 0$ this implies $c_B = 0$ for all $B \in \mathcal{A}$. If $\lambda \neq 0$ then c_B has the opposite sign to $\sum_{A \in \mathcal{A}} c_A$, a contradiction unless $c_B = 0$ for all $B \in \mathcal{A}$.

Either way, the x_A are linearly independent so $|\mathcal{A}| \leq n$. □

What if we allow more than one intersection size?

Theorem 5.2. Let $\mathcal{L} \subset \mathbb{N} \cup \{0\}$ and let $\mathcal{A} \subset [n]^{(r)}$ be such that $|A \cap B| \in \mathcal{L}$ for distinct $A, B \in \mathcal{A}$. Suppose $\gcd(\mathcal{L}) \nmid r$. Then $|\mathcal{A}| \leq n$.

Proof. Let $x_A \in \mathbb{Q}^n$ be the characteristic vector of $A \in \mathcal{A}$. Then there exists integers j_A with $\sum_{A \in \mathcal{A}} j_A x_A = 0$ with $\gcd\{j_A : A \in \mathcal{A}\} = 1$.

Take a prime power p^k with $p^k \mid l$ for all $l \in \mathcal{L}$ but $p^k \nmid r$. Dotting with x_B gives

$$\sum_{A \in \mathcal{A}} j_A |A \cap B| = 0.$$

Hence $p^k \mid j_B|B|$ for all B. Thus $p \mid j_B$, contradicting $\gcd\{j_A : A \in \mathcal{A}\} = 1$. □

More generally, $\mathcal{A}$ can be bigger. If all we know is that $|\mathcal{L}| = s$ then $[n]^{(s)}$ and $[n]^{(\leq s)}$ are examples of uniform and non-uniform families of sizes $\binom{n}{s}$ and $\binom{n}{0}+\cdots+\binom{n}{s}$, respectively. These bounds are in fact tight: proved by Ray–Chaudhuri and Wilson (uniform case, 1975) and Babai (non-uniform case, 1980's).

Theorem 5.3. Let $\mathcal{L} \subset \mathbb{N} \cup \{0\}$, $|\mathcal{L}| = s$. Let $\mathcal{A} \subset \mathcal{P}[n]$ with $|A \cap B| \in \mathcal{L}$ for distinct $A, B \in \mathcal{A}$. Then

$$|\mathcal{A}| \leq \binom{n}{0} + \cdots + \binom{n}{s}.$$

Proof. For $A \in \mathcal{A}$ define the polynomial $f_A\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f_A(x) = \prod_{\substack{l \in \mathcal{L} \\ l < |A|}} (\langle x, x_A \rangle - l)$$

where $x = (x_1, \ldots, x_n)$ and $\langle x, x_A \rangle = \delta_1 x_1 + \cdots + \delta_n x_n$. Note $f_A(x_B) = 0$ unless $A \subset B$. Let $\tilde{f}_A$ be the polynomial obtained from f_A by replacing all powers x_i^e, $e \geq 2$, by x_i. Then $\tilde{f}_A(x_B) = 0$ unless $A \subset B$, because $f_A(x_B) = \tilde{f}_A(x_B)$ as $x_B \in \{0,1\}^n$.

Suppose $\sum_{A \in \mathcal{A}} c_A \tilde{f}_A(x) = 0$. Pick B with $|B|$ minimal and $c_B \neq 0$. Then

$$\sum_{A \in \mathcal{A}} c_A \tilde{f}_A(x_B) = c_B \tilde{f}_B(x_B) \neq 0$$

a contradiction. So the $\tilde{f}_A$ are linearly independent.

The $\tilde{f}_A$ are spanned by all monomials $\prod_{i \in T} x_i$ where $T \subset [n]^{(\leq s)}$. Hence

$$|\mathcal{A}| \leq |[n]^{(\leq s)}|$$

□

A long-standing conjecture that $|\mathcal{A}| \leq \sum_{i=0}^{s} \binom{n-1}{i}$ if $0 \notin \mathcal{L}$, achievable by $\{Y \in [n]^{(\leq s+1)} : 1 \in Y\}$, was proved by Snerily (2003). Compare this with Fisher's Inequality.

Why not replace the underlying field in the proof by $GL(p)$?

We use the notation $m \in \mathcal{L} \pmod p$ if there exists $l \in \mathcal{L}$ with $m \equiv l \pmod p$, $m \notin \mathcal{L} \pmod p$ means that for all $l \in \mathcal{L}$ we have $m \not\equiv l \pmod p$.

Theorem 5.4. Let p be a prime. Let $\mathcal{L} \subset \mathbb{N} \cup \{0\}$, $|\mathcal{L}| = s$. Let $\mathcal{A} \subset \mathcal{P}[n]$ such that $|A| \notin \mathcal{L} \pmod p$ for all $A \in \mathcal{A}$ but $|A \cap B| \in \mathcal{L} \pmod p$ for distinct $A, B \in \mathcal{A}$. Then

$$|\mathcal{A}| \leq \binom{n}{0} + \cdots + \binom{n}{s}.$$

Proof. Repeat the proof of Theorem 5.3 but over $GL(p)$ instead of $\mathbb{R}$ and with

$$f_A(x) = \prod_{l \in \mathcal{L}} (\langle x_A, x \rangle - l).$$

Then $f_A(x_A) \neq 0$ but $f_A(x_B) = 0$ for all $B \neq A$ so the f_A (and the $\tilde{f}_A$) are linearly independent. □

The following "uniform" version uses a different proof.

Theorem 5.5 (Frankl–Wilson, 1981)**.** Let p be a prime number, let $\mathcal{L} \subset \mathbb{N} \cup \{0\}$, $|\mathcal{L}| = s$, suppose $r \in \mathbb{N}$ is such that $r \notin \mathcal{L} \pmod p$. Let $\mathcal{A} \subset \mathcal{P}[n]$ be such that $|A| \equiv r \pmod p$ for all $A \in \mathcal{A}$ and $|A \cap B| \in \mathcal{L} \pmod p$ for distinct $A, B \in \mathcal{A}$.

Suppose moreover that $r \notin \{0, 1, \dots, s-1\} \pmod p$. Then

$$|\mathcal{A}| \le \binom{n}{s}.$$

Remark. The condition $r \notin \{0, 1, \dots, s-1\} \pmod p$ is an artefact of the proof. With more work it can be replaced by say $r + s < n$.

Proof. Let $\mathcal{A}_i$ and M_i, $0 \le i \le s$, be the $|\mathcal{A}| \times \binom{n}{i}$ and the $\binom{n}{s} \times \binom{n}{i}$ matrices whose rows are indexed by $\mathcal{A}$ and $[n]^{(s)}$ and whose columns are indexed by $[n]^{(i)}$. The entry in row A and column B is 1 if $A \supset B$ and 0 if $A \not\supset B$.

Let V be the vector space over $GL(p)$ spanned by the columns of $\mathcal{A}_s$. Then $\dim V \le \binom{n}{s}$. Note that if M is any $\binom{n}{s} \times t$ matrix then the columns of $\mathcal{A}_s M$ are in V.

Let $A \in \mathcal{A}$ and $I \in [n]^{(i)}$. Then

$$(\mathcal{A}_s M_i)_{AI} = |\{S \in [n]^{(s)} : A \supset S \supset I\}|$$

which is 0 if $I \not\subset A$ and $\binom{|A|-i}{s-i}$ if $I \subset A$. Thus $\mathcal{A}_s M_i \equiv \binom{r-i}{s-i} \mathcal{A}_i \pmod p$. By the condition, $\binom{r-i}{s-i} \not\equiv 0 \pmod p$, so the columns of $\mathcal{A}_i$ are in V.

Thus the columns of $\mathcal{B}_i = \mathcal{A}_i \mathcal{A}_i^T$ are in V. This is an $|\mathcal{A}| \times |\mathcal{A}|$ matrix whose (A, B) entry is $|\{I \in [n]^{(i)} : I \subset A, I \subset B\}| = \binom{|A \cap B|}{i}$.

Consider the polynomial

$$\phi(x) = \prod_{l \in \mathcal{L}} (x - l).$$

Then $\phi(r) \ne 0$ but $\phi(l) = 0$ for all $l \in \mathcal{L}$ over $GF(p)$. Take scalars $c_0, \dots, c_s$ so

$$\phi(x) = c_0 \binom{x}{0} + \dots + c_s \binom{x}{s}.$$

Let $\mathcal{B} = c_0 \mathcal{B}_0 + \dots + c_s \mathcal{B}_s$. The (A, B) entry of $\mathcal{B}$ is $\phi(|A \cap B|)$. So $\mathcal{B}$ is zero $\pmod p$ off-diagonal and non-zero on-diagonal. So $\mathcal{B}$ is non-singular and its columns are in V. Hence

$$|\mathcal{A}| = \operatorname{rank} \mathcal{B} \le \dim V \le \binom{n}{s}.$$ □

Corollary 5.6 (Ray–Chaudhuri, Wilson, 1975)**.** Let $\mathcal{A} \subset [n]^{(r)}$ and let $\mathcal{L} = \{|A \cap B| : A, B \in \mathcal{A}, A \ne B\}$. Then

$$|\mathcal{A}| \le \binom{n}{|\mathcal{L}|}.$$

Proof. Clearly, $r \notin \mathcal{L}$ and $r \ge s = |\mathcal{L}|$. Choose a prime greater than r and apply Theorem 5.5. □

We might ask whether Theorem 5.5 holds for non-primes p. Here is a special case.

Theorem 5.7. Let $q < r$ be a prime power. Let $\mathcal{A} \subset [n]^{(r)}$ be such that $|A \cap B| \not\equiv r \pmod{q}$ for distinct $A, B \in \mathcal{A}$. Then

$$|\mathcal{A}| \leq \binom{n}{q-1}.$$

Proof. Copy the previous proof but work over $\mathbb{Q}$. Then $\mathcal{A}_s M_i = \binom{r-i}{s-i}\mathcal{A}_i$ so the columns of $\mathcal{A}_i$ are in V over $\mathbb{Q}$. Let $\phi(x) = \binom{r-1-x}{q-1}$ and choose $c_0, \ldots, c_s$ with

$$\phi(x) = \sum_{i=0}^{s} c_i \binom{x}{i}$$

where $s = q - 1$. Then the (A, B) entry in $\mathcal{B}$ is $\phi(|A \cap B|)$.

If $A = B$ the entry is $\phi(r) = \binom{-1}{q-1} = (-1)^{q-1} \not\equiv 0 \pmod{p}$ where q is a power of p. On the other hand, the identity

$$(r-l)\phi(l) = q\binom{r-l}{q}$$

is an identity in four integers and if $r - l \not\equiv 0 \pmod{q}$ then $\phi(l) \equiv 0 \pmod{p}$. So $\mathcal{B}$ is non-singular. $\square$

Does Frankl–Wilson hold for non-prime moduli? Do we still obtain a polynomial bound, if s is fixed?

Grolmusz (2000) gave examples where this fails. There exists a uniform family $\mathcal{A} \subset [n]^{(r)}$ where $r \equiv 0 \pmod{6}$ and $|A \cap B| \not\equiv 0 \pmod{6}$ for distinct $A, B \in \mathcal{A}$ and

$$|\mathcal{A}| = \exp\left\{\left(\tfrac{1}{27} + o(1)\right)\frac{\log^2 n}{\log\log n}\right\}$$

In this example, $r \approx n^{1-\frac{1}{27}}$.

Chapter 6

Breathtaking Consequences

The graph on $\mathbb{R}^n$ has points of $\mathbb{R}^n$ as vertices and edges joining points at distance 1. What is $\chi(\mathbb{R}^n)$? If $n = 2$, we know $4 \le \chi(\mathbb{R}^2) \le 7$. In general, $\chi(\mathbb{R}^n) \le 3^n$ by tiling $\mathbb{R}^n$ with cubes. A compactness argument by Erdős and de Bruyn shows that there exist a finite subgraph H with $\chi(H) = \chi(\mathbb{R}^n)$.

Corollary 6.1 (Frankl–Wilson, 1981)**.**

$$\chi(\mathbb{R}^n) \ge (1.2 + o(1))^n.$$

Proof. Let G be the subgraph spanned by

$$V = \left\{ \frac{1}{\sqrt{2q}} x_A : A \subset [n]^{(2q-1)} \right\}.$$

Then two points in V are distance 1 apart if and only if $|A \cap B| = q - 1$. If q is a prime or a prime power then no colour can be used more than $\binom{n}{q-1}$ times by Theorem 5.7. Hence

$$\chi(G) \ge \frac{\binom{n}{2q-1}}{\binom{n}{q-1}}.$$

Take $q = (2 - \sqrt{2} + o(1))\frac{n}{4}$ and q prime. □

Even older and more famous is *Borsuk's Conjecture*: every set of diameter 1 in $\mathbb{R}^n$ is the union of $n + 1$ sets of diameter less than 1. This is true if $n = 3$, and if the sets are smooth or centrally symmetric.

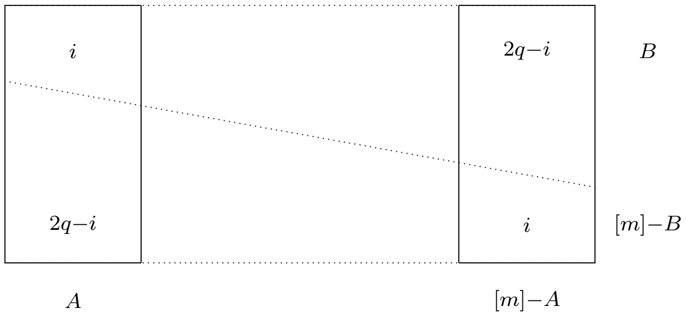
Corollary 6.2 (Kahn–Kalai, 1993)**.** There is a set in $\mathbb{R}^n$ of diameter 1 that is not the union of $1.2^{\sqrt{n}}$ sets of diameter less than 1.

Proof. Choose $m = 4q$ with $\binom{m}{2} \approx n$. Think of the coordinates of $\mathbb{R}^n$ as being the edges of the complete graph $[m]^{(2)}$.

For each subset $A \in [m]^{(m/2)}$ let v_A be the characteristic vector of the edges between A and $[m] - A$. So v_A and $v_{[m]-A}$ are equal; they have $4q^2$ ones. If A, B are two subsets then

$$d^2(v_A, v_B) = 4i(2q - i)$$

where $i = |A \cap B|$.



Thus $d^2(v_A, v_B) \leq 4q^2$ with equality if and only if $|A \cap B| = q$. Let

$$S = \left\{ \frac{1}{2q} v_A : 1 \in A \in [m]^{(m/2)} \right\}.$$

Then $\operatorname{diam}(S) = 1$ and $|S| = \frac{1}{2}\binom{m}{m/2}$. But if $T \subset S$ has diameter less than 1 then $|A \cap B| \neq q$ for $v_A, v_B \in 2qT$. Let

$$\mathcal{A} = \{A - 1 : v_A \in 2qT\}.$$

Then $|A' \cap B'| \neq q-1$ for $A', B' \in \mathcal{A}$. Also $\mathcal{A} \subset [m]^{(2q-1)}$. By Theorem 5.7, $|T| = |\mathcal{A}| \leq \binom{m-1}{q-1}$. Then

$$\begin{aligned} \frac{|S|}{|T|} &\geq \frac{\frac{1}{2}\binom{m}{m/2}}{\binom{m-1}{q-1}} \\ &= \frac{2\binom{m}{m/2}}{\binom{m}{m/4}} \\ &\geq (1.14 + o(1))^m \\ &\geq 1.2^{\sqrt{n}} \end{aligned}$$

□

Recall that the simplest non-trivial case of Ramsey's theorem asserts the existence of a number $R(t)$, the smallest n such that every colouring of the edges of K_n with red and blue yields a monochromatic K_t.

It is easily shown that

$$R(t) \leq \binom{2t-2}{t-1} \leq 2^{2t}.$$

Erdős showed $\sqrt{2}^t$ by means of an existential proof. We would like an explicit colouring. It is trivial to colour $K_{(t-1)^2}$ with no monochromatic K_t so $R(t) \geq (t-1)^2$ constructively. But better polynomial bounds can be achieved.

Achieving super-polynomial bounds is trickier. Let G be the colouring of K_N where $N = \binom{n}{r}$ where vertices are $[n]^{(r)}$ and AB is red if $|A \cap B| \not\equiv -1 \pmod{q}$. There q is a prime power and $r > q$ is chosen with $r \equiv -1 \pmod{q}$. Theorem 5.7 shows G has no red K_t if $t > \binom{n}{q-1}$. If we take $r = q^2 - 1$ then by Corollary 5.6 there is no blue K_t if $t > \binom{n}{q-1}$.

Corollary 6.3. This construction shows

$$R(t) \geq \exp\left\{\left(\frac{1}{4} + o(1)\right)\frac{\log^2 t}{\log\log t}\right\}.$$

Proof. Take $n = q^3$. □

This remains the best known "construction".

Remark. We could have used our non-uniform bounds to obtain essentially the same result. The following example is due to Alon. Let p, q be two primes, $r = pq - 1$, $N = \binom{n}{r}$. Colour K_N on $[n]^{(r)}$ by AB red if $|A \cap B| \not\equiv -1 \pmod{q}$. Then there is no red K_t with $t > \binom{n}{q-1} + \cdots + \binom{n}{0}$ by Theorem 5.4 or with $t > \binom{n}{q-1}$ by Theorem 5.5. There is no blue K_t if $t > \binom{n}{p-1} + \cdots + \binom{n}{0}$ by Theorem 5.3 or $t > \binom{n}{p-1}$ by Corollary 5.6.

The bipartite Ramsey number $BR(t)$ is the smallest N such that every red and blue colouring or the edges of the complete bipartite graph $K_{N,N}$ contains a monochromatic $K_{t,t}$. It is easy to show that

$$\sqrt{2}^t < BR(t) \leq t2^t.$$

A "good" colouring of $K_{N,N}$ yields a "good" colouring of K_N by identification of pairs, but the converse fails.

Until recently, the best known bipartite colouring was trivial $BR(t) \geq t^2$. Barak–Rao–Shealtiel–Wigderson (2006) showed

$$BR(t) \geq \exp\left((\log t)^{\omega(t)}\right)$$

where $\omega(t) \to \infty$. The proof gives an algorithm which decides, for each edge in $K_{N,N}$, whether to colour it red or blue in polynomial time. Is this a construction?

Chapter 7

Shannon Capacity

We wish to transmit messages over some channel using an alphabet V, where some pairs of letters can get confused. The *confusion graph* is the graph G with vertex set V where $ab \in E(G)$ if a can be confused with b.

An *independent set* in a graph is a set of vertices spanning no edge. The *independence number* $\alpha(G)$ is the maximum size of an independent set. So our effective alphabet size is $\alpha(G)$.

If we have a memory of up to n letters, we are more interested in $\alpha(G^n)$ where $V(G^n) = V(G)^n$ and

$$E(G^n) = \{(a_1, \dots, a_n)(b_1, \dots, b_n) : a_i = b_i \text{ or } a_i b_i \in E(G) \text{ for all } i\}.$$

We can send $\alpha(G^n)$ messages of length n.

Note if $U \subset V(G)^k$ is independent in G^k and $W \subset V(G)^l$ is independent in G^l then $U \times W$ is independent in G^{k+l}. So

$$\alpha(G^{k+l}) \geq \alpha(G^k)\alpha(G^l).$$

By Fekete's lemma, or superadditivity,

$$\lim_{n\to\infty} \alpha(G^n)^{1/n} = \sup_{n\in\mathbb{N}} \alpha(G^n)^{1/n}$$

exists.

Definition. The *Shannon capacity* of G is

$$c(G) = \lim_{n\to\infty} \alpha(G^n)^{1/n}.$$

It is the effective alphabet size for long messages.

Example. Let $G = C_5$ on $[5]$. Then $\alpha(G) = 2$ and

$$\{(1,1), (2,3), (3,5), (4,2), (5,4)\}$$

is independent in G^2, so $\alpha(G^2) \geq 5$.

Finding capacities with error correction comes down to finding the Shannon capacity of another graph.

Suppose we have two disjoint channels with confusion graphs G, H. This corresponds to one channel with confusion graph $G \sqcup H$ where $V(G \sqcup H) = V(G) \sqcup V(H)$ and $E(G \sqcup H) = E(G) \sqcup E(H)$.

Lemma 7.1.

$$c(G \sqcup H) \geq c(G) + c(H).$$

Proof. Exercise. □

Shannon (1956) conjectured that equality holds. Alon (1998) showed the conjecture is utterly false. A crucial decision is to take $H = \bar{G}$, because of the next lemma.

Lemma 7.2. Let $n = |G| = |\bar{G}|$ then

$$c(G \sqcup \bar{G}) \geq \sqrt{2n}.$$

Proof. Label G as $a_1, \ldots, a_n$ and $\bar{G}$ as $b_1, \ldots, b_n$ where $a_i a_j \in E(G)$ if and only if $b_i b_j \notin E(\bar{G})$. Then

$$\{(a_i, b_i) : 1 \leq i \leq n\} \cup \{(b_i, a_i) : 1 \leq i \leq n\}$$

is an independent set of size $2n$ in $(G \sqcup \bar{G})^2$ so

$$c(G \sqcup \bar{G}) \geq \sqrt{\alpha((G \sqcup \bar{G})^2)} \geq \sqrt{2n}.$$ □

To find a counterexample we need a graph G with both $\alpha(G)$ and $\alpha(\bar{G})$ small. This is the Ramsey problem and random graphs give excellent examples. But no-one can bound the capacity of random graphs sufficiently from above. Our construction of Ramsey graphs might work. The simplest construction was based on Theorem 5.3.

Let F be a field and M a subspace of $F[X_1, \ldots, X_r]$, the space of polynomials in r variables over F. A *representation* of a graph G over M is an assignment to each vertex $u \in V(G)$ of a polynomial $f_u \in M$ and a vector $c_u \in F^r$ such that $f_u(c_u) \neq 0$ and $f_u(c_v) = 0$ for all distinct $u, v \in V(G)$ with $uv \notin E(G)$.

Lemma 7.3. If G has a representation over M then $\alpha(G) \leq \dim M$.

Proof. If $U = \{u_1, \ldots, u_\alpha\}$ is an independent set in G, and if

$$\sum \lambda_i f_{u_i} = 0$$

then

$$\sum \lambda_i f_{u_i}(c_{u_j}) = \lambda_j f_{u_j}(c_{u_j})$$

so $\lambda_j = 0$ for all j. So $\{f_{u_1}, \ldots, f_{u_\alpha}\}$ is a linearly independent subset of M. □

The usefulness of this idea hangs on the next lemma.

Lemma 7.4. If G has a representation over M and H has a representation over N, both over the same field F, then $G \cdot H$ has a representation over $M \otimes N$, so

$$\alpha(G \cdot H) \leq \dim M \dim N.$$

Here $G \cdot H$ has vertex set $V(G) \times V(H)$ and

$$E(G) = \{(a, b)(a', b') : a' = a \text{ or } aa' \in E(G),\ b' = b \text{ or } bb' \in E(H)\}.$$

Proof. Let $M \subset F[X_1, \ldots, X_r]$ and $N \subset F[Y_1, \ldots, Y_s]$. Let $\{g_u, c_u : u \in V(G)\}$ represent G and $\{h_v, d_v : v \in V(H)\}$ represent H. For $(u, v) \in V(G \cdot H)$ let

$$f_{(u,v)}(X_1, \ldots, X_r, Y_1, \ldots, Y_s) = g_u(X_1, \ldots, X_r)h_v(Y_1, \ldots, Y_s).$$

Clearly the polynomials $f_{(u,v)}$ lie in $F[X_1, \ldots, X_r, Y_1, \ldots, Y_s]$, in a subspace of dimension at most $\dim M \dim N$.

Moreover $(c_u, d_v) \in F^{r+s}$. Then the set

$$\{f_{(u,v)}, (c_u, d_v) : (u, v) \in V(G \cdot H)\}$$

represents $G \cdot H$ since $f_{(u,v)}(c_{u'}, d_{v'}) = g_u(c_{u'})h_v(d_{v'})$ and this is not 0 if $(u, v) = (u', v')$ but 0 if $(u, v)(u', v') \notin E(G \cdot H)$. □

Corollary 7.5. If G has a representation over M then

$$c(G) \leq \dim M.$$

Proof. By Lemma 7.4, $\alpha(G^n) \leq (\dim M)^n$. □

How can we apply this to $G \sqcup \bar{G}$? Note that $G \cdot \bar{G}$ is a graph of order $|G|^2$ with an independent set of size $|G|$, so Lemma 7.4 says if we can represent $G, \bar{G}$ by M, N *over the same field* F then $\dim M \dim N \geq n$ so $\dim M + \dim N \geq 2\sqrt{n}$, so we cannot disprove Shannon's conjecture this way.

Try different fields. Let p, q be distinct primes and $r = pq - 1$. Let G be the graph on vertex set $[n]^{(r)}$ where $AB \in E(G)$ if $|A \cap B| \equiv -1 \pmod p$. Let M be the space of multilinear polynomials in variables $X_1, \ldots, X_n$ of total degree at most $p - 1$ over $GF(p)$. Then

$$\dim M \leq \binom{n}{0} + \cdots + \binom{n}{p-1}.$$

Let N be the corresponding space with p replaced by q.

Lemma 7.6. G is representable over M.

Proof. For $A \in [n]^{(r)}$ let x_A be its characteristic vector and

$$f_A(x) = \prod_{j=0}^{p-2} (\langle x, x_A \rangle - j)$$

over $GF(p)$. Let $\tilde{f}_A \in M$ be as in the proof of Theorem 5.3. Then $\{\tilde{f}_A, x_A : A \in V(G)\}$ represents G over M. □

Lemma 7.7. $\bar{G}$ is representable over N.

Proof. Let

$$g_A(x) = \prod_{j=0}^{q-2} (\langle x, x_A \rangle - j)$$

over $GF(q)$. If $AB \notin E(\bar{G})$ then $|A \cap B| \equiv -1 \pmod p$; thus $|A \cap B| \not\equiv -1 \pmod q$, else $|A \cap B| \equiv -1 \pmod{pq}$ so $A = B$. Thus $\{\tilde{g}_A, x_A : A \in [n]^{(r)}\}$ represents $\bar{G}$ over N. □

Theorem 7.8 (Alon, 1998)**.** For each $t \in \mathbb{N}$ there exists a graph G with $c(G), c(\bar{G}) \leq t$ and

$$c(G \sqcup \bar{G}) \geq \exp\left\{\left(\frac{1}{8} + o(1)\right)\frac{\log^2 t}{\log\log t}\right\}$$

where $o(1) \to 0$ as $t \to \infty$.

Proof. Pick primes p, q with $q < p < q + o(q)$ and let $n = p^3$. Let G be the graph just constructed. Then

$$c(G), c(\bar{G}) \leq \sum_{j=0}^{p-1}\binom{n}{j} = \dim M \leq 2\binom{n}{p-1}$$

by Lemma 7.6, Lemma 7.7 and Corollary 7.5. On the other hand,

$$c(G \sqcup \bar{G}) \geq \sqrt{2\binom{n}{pq-1}}$$

by Lemma 7.2 □

Chapter 8

The Lovász θ Function

An *orthonormal representation (ONR)* of a graph G is a collection of unit vectors in $\mathbb{R}^k$, some k, one for each vertex of G, such that non-adjacent vertices have orthogonal vectors.

The *tensor product* of $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^l$ is the vector $u \otimes v$ in $\mathbb{R}^{kl}$; co-ordinate-wise, if $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_l)$ then

$$u \otimes v = (u_1 v_1, u_2 v_1, \ldots, u_k v_1, u_1 v_2, u_2 v_2, \ldots, u_k, v_2, \ldots, u_1 v_l, u_2 v_l, \ldots, u_k v_l).$$

Notice $\langle u \otimes v, u' \otimes v' \rangle = \langle u, u' \rangle \langle v, v' \rangle$. So if G has ONR $\{u_i\}$ and H has ONR $\{v_j\}$ then $G \cdot H$ has ONR $\{u_i \otimes v_j\}$.

Thus, similar to before, if G is representable over $\mathbb{R}^k$ then $\alpha(G) \leq k$, $\alpha(G^n) \leq k^n$ so $c(G) \leq k$.

We can do better if all the vectors in the representation lie in similar directions. The *value* of a representation is

$$\operatorname{val}\{u_i\} = \min_{c \in \mathbb{R}^k, \|c\|=1} \max_{i \in V(G)} \frac{1}{\langle c, u_i \rangle^2}.$$

A vector c attaining this minimum is called a *handle*.

Definition.

$$\theta(G) = \min\{\operatorname{val}\{u_i\} : \{u_i\} \text{ represents } G\}.$$

Lemma 8.1.

$$\alpha(G) \leq \theta(G).$$

Proof. If $\{u_i\}$ is an ONR and $W \subset V(G)$ is independent, then $\{u_i : i \in W\}$ is an orthogonal system, so

$$\frac{|W|}{\operatorname{val}\{u_i\}} \leq \sum_{i \in W} \langle c, u_i \rangle^2 \leq |c|^2 = 1. \qquad \square$$

Lemma 8.2.

$$\theta(G \cdot H) \leq \theta(G)\theta(H).$$

Proof. Take ONRs $\{u_i\}, \{v_j\}$ for G, H together with handles c, d.

$$\theta(G) = \max_i \frac{1}{\langle c, u_i \rangle^2}, \qquad \theta(H) = \max_j \frac{1}{\langle d, v_j \rangle^2}$$

Now $\langle u_i \otimes v_j, u_l \otimes v_m \rangle = \langle u_i, u_l \rangle \langle v_j, v_m \rangle$ so $\{u_i \otimes v_j\}$ is an ONR of $G \cdot H$. Let $e = c \otimes d$. Then $\langle e, e \rangle = \langle c, c \rangle \langle d, d \rangle = 1$ so e is a unit vector. Then

$$\theta(G \cdot H) \leq \max_{i,j} \frac{1}{\langle e, u_i \otimes v_j \rangle^2} = \max_{i,j} \frac{1}{\langle c, u_i \rangle^2} \frac{1}{\langle d, v_j \rangle^2} = \theta(G)\theta(H). \qquad \square$$

Theorem 8.3. For any G, $c(G) \leq \theta(G)$.

Proof. By Lemma 8.1, $\alpha(G^n) \leq \theta(G^n)$. By Lemma 8.2, $\theta(G^n) \leq \theta(G)^n$. □

For over 20 years no-one knew $c(C_5)$. The best bounds were $\sqrt{5} \leq c(C_5) \leq 3$.

Corollary 8.4 (Lovász, 1979)**.**

$$c(C_5) = \sqrt{5}.$$

Proof. Consider an umbrella with handle c and with five spokes. Gradually open it till alternate spokes are orthogonal. It is easily checked that

$$\langle c, u_i \rangle^2 = \frac{1}{\sqrt{5}}$$

so $\theta(C_5) \leq \sqrt{5}$. □